Massive Particle Model with Spin from a Hybrid (spacetime-twistorial) Phase Space Geometry and Its Quantization

Sergey Fedoruk*, Andrzej Frydryszak†, Jerzy Lukierski‡ and Cèsar Miquel-Espanya§

Abstract

We extend the Shirafuji model for massless particles with primary spacetime coordinates and composite four-momenta to a model for massive particles with spin and electric charge. The primary variables in the model are the spacetime four-vector, four scalars describing spin and charge degrees of freedom as well as a pair of Weyl spinors. The geometric description proposed in this paper provides an intermediate step between the free purely twistorial model in two-twistor space in which both spacetime and four-momenta vectors are composite, and the standard particle model, where both spacetime and four-momenta vectors are elementary. We quantize the model and find explicitly the first-quantized wavefunctions describing relativistic particles with mass, spin and electric charge. The spacetime coordinates in the model are not commutative; this leads to a wavefunction that depends only on one covariant projection of the spacetime four-vector defining plane wave solutions.

1 Introduction

There are three known equivalent ways of describing massless relativistic particles:

1. *Purely twistorial description* - with primary twistor variables and composite both spacetime and four-momenta [1];

*BLTP JINR, Dubna, Russia e-mail: fedoruk@theor.jinr.ru
†ITP University of Wrocław, Poland e-mail: amfry@ift.uni.wroc.pl
‡ITP University of Wrocław, Poland e-mail: lukier@ift.uni.wroc.pl
§Dept. de Física Teòrica and IFIC, València, Spain & C.N. YITP, Stony Brook, USA e-mail: Cesar.Miquel@ific.uv.es

2. *Mixed twistorial-spacetime (Shirafuji-like) description* - with primary spacetime coordinates and composite four-momenta [2]; and

3. *Standard geometric description* - with primary relativistic phase space variables (spacetime coordinates and four-momenta).

The extension of these three geometric levels to the two-twistor sector has been presented recently [3, 4, 5] in terms of the corresponding Liouville one-forms. If we introduce two twistors ($i = 1, 2;\ A = 1, \ldots, 4$)[1]

$$Z_{Ai} = ({\omega^\alpha}_i, \bar{\pi}_{\dot{\alpha}i}) \,,$$

the free Liouville one-form corresponding to the two-twistor case is the following

$$\Theta_2 = \frac{i}{2} \left({\omega^\alpha}_i d\pi_{\alpha i} + \bar{\pi}_{\dot{\alpha}i} d\bar{\omega}^{\dot{\alpha}i} - h.c. \right) \,.$$

After using the two-twistor generalization of the Penrose incidence relation and the realization of the momentum $P_{\alpha\dot{\alpha}}$,

$$P_{\alpha\dot{\beta}} = \pi^i_\alpha \bar{\pi}_{\dot{\beta}i} \,, \qquad {\omega^\alpha}_i = i z^{\alpha\dot{\beta}} \bar{\pi}_{\dot{\beta}i} \,, \qquad \text{where} \quad z^{\alpha\dot{\beta}} = x^{\alpha\dot{\beta}} + i y^{\alpha\dot{\beta}} \,,$$

one obtains $\Theta_2' = {\pi_\alpha}^i \bar{\pi}_{\dot{\beta}i} dx^{\alpha\dot{\beta}} + i y^{\alpha\dot{\beta}} ({\pi_\alpha}^i d\bar{\pi}_{\dot{\beta}i} - \bar{\pi}_{\dot{\beta}i} d{\pi_\alpha}^i)$.

We introduce new variables ${s^i}_j = -2 y^{\alpha\dot{\beta}} \pi^i_\alpha \bar{\pi}_{\dot{\beta}j} = \overline{({s^j}_i)}$, and define f and $\overline{f}$ satisfying

$$\begin{array}{lclclc} \bar{\pi}^{\dot{\alpha}i} {\bar{\pi}_{\dot{\alpha}}}^{\ j} = -\epsilon^{ij} f & , & & {\pi^\alpha}_i \pi_{\alpha j} = -\epsilon_{ij} \bar{f} & , \\ \bar{\pi}_{\dot{\alpha}i} {\bar{\pi}_{\dot{\beta}}}^{\ i} = \epsilon_{\dot{\alpha}\dot{\beta}} f & , & & \pi_{\alpha i} {\pi_\beta}^i = \epsilon_{\alpha\beta} \bar{f} \,. & \end{array}$$

Now, we can write

$$\Theta_2' = {\pi_\alpha}^i \bar{\pi}_{\dot{\alpha}i} dx^{\alpha\dot{\alpha}} + \frac{i}{2} {s_i}^j \left[\frac{1}{\bar{f}} {\pi^\alpha}_k d\pi_{\alpha j} \epsilon^{ki} + \frac{1}{f} \bar{\pi}^{\dot{\alpha}i} d{\bar{\pi}_{\dot{\alpha}}}^{\ k} \epsilon_{kj} \right] \,. \tag{1}$$

Formula (1) determines the two-twistor generalization of the Shirafuji action [2]. We see that the primary or, equivalently, elementary variables are now the following ones

$$\begin{array}{rcl} N = 1 & \Rightarrow & N = 2 \\ x^{\alpha\dot{\alpha}}, \pi_\alpha, \bar{\pi}_{\dot{\alpha}} & \Rightarrow & x^{\alpha\dot{\alpha}}, \pi_{\alpha i}, {\bar{\pi}_{\dot{\alpha}}}^{\ i}, {s_i}^j \,. \end{array}$$

[1] The indices $i = 1, 2$ describe an internal $SU(2)$ symmetry. The complex conjugation implies the change from covariant (lower) indices to contravariant (upper) indices.

The particle model described by the Liouville one-form (1) provides a framework to describe the mass, spin and electric charge but does not specify their values. We shall introduce further their numerical values by postulating suitable physical constraints. One concludes that the quantum-mechanical solution of the model (1) may describe infinite-dimensional higher spin and electric charge multiplets, linked with the field-theoretic formulation of higher spin theories (see e.g. [6, 7, 8]).

2 The classical model - analysis of constraints in phase space

2.1 Action, conservation laws and physical constraints

The dynamics of a massive spinning particle is described by its trajectory in the generalized coordinate space

$$Q_L(\tau) = \left(x^\mu(\tau), \pi_{\alpha k}(\tau), \bar{\pi}^k_{\dot{\alpha}}(\tau), {s_k}^j(\tau) \right) , \tag{2}$$

where $\pi_{\alpha k}$, $\bar{\pi}^k_{\dot{\alpha}} = \overline{(\pi_{\alpha k})}$ $(k, j = 1, 2)$ are two pairs of commuting Weyl spinors and the four quantities ${s_k}^j$, satisfying the condition ${s_k}^j = \overline{({s_j}^k)}$, are Lorentz scalars. The action derived from (1) has the following form $(a = 1, \ldots, 4)$

$$S = \int d\tau \, \mathcal{L} = \int d\tau \, \left[{\pi_\alpha}^i \bar{\pi}_{\dot{\alpha} i} \dot{x}^{\alpha\dot{\alpha}} + \frac{i}{2} {s_k}^j \left(\frac{1}{\bar{f}} \pi^{\alpha k} \dot{\pi}_{\alpha j} + \frac{1}{f} \bar{\pi}^{\dot{\alpha}}_j \dot{\bar{\pi}}^k_{\dot{\alpha}} \right) + \lambda^a T_a \right] , \tag{3}$$

where the T_a are four algebraic constraints on the coordinates (2) to be specified later and λ^a are their Lagrange multipliers.

From the Lagrangian (3) we obtain the canonical momenta in the standard way ($\mathcal{P}^L = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{Q}}_L}$). This leads to the following 16 primary constraints

$$P^{\alpha\dot{\alpha}} - \pi^{\alpha k} \bar{\pi}^{\dot{\alpha}}_k \approx 0 \, , \quad {P_{(s)}}^k{}_j \approx 0 \, , \tag{4}$$

$$P^{\alpha j} - \frac{i}{2\bar{f}} \pi^{\alpha k} {s_k}^j \approx 0 \quad , \qquad \bar{P}^{\dot{\alpha}}_j - \frac{i}{2f} {s_j}^k \bar{\pi}^{\dot{\alpha}}_k \approx 0 \, , \tag{5}$$

We present now the form of the four algebraic constraints[2] T_a

[2]The justification of the form of the constraints T_a can be obtained by considering the symmetries of the action (3). It appears that the choice (6)-(9) and the interpretation of s_i as covariant spin projection is related with the formulae for the corresponding Noether charges.

$$T_1 : \qquad T \equiv 4f\bar{f} - m^2 \approx 0\,, \tag{6}$$

$$T_2 : \qquad S \equiv \mathbf{s}^2 - s(s+1) \approx 0\,, \tag{7}$$

$$T_3 : \qquad S_3 \equiv s_3 - m_3 \approx 0\,, \tag{8}$$

$$T_4 : \qquad Q \equiv s_0 - q \approx 0\,. \tag{9}$$

The real quantities $\mathbf{s} = (s_r) = (s_1, s_2, s_3)$ and s_0 present in (7)-(9) are defined in terms of the Lagrangian variables s_k^j as follows

$$s_0 = \frac{1}{2} s_k{}^k\,, \qquad s_r = \frac{1}{2} s_k{}^j (\sigma_r)_j{}^k\,, \qquad r = 1,2,3\,,$$

where $(\sigma_r)_j{}^k$ are the Pauli matrices.

The constraint (6) defines the mass m of the particle because using it we obtain that ($P_\mu \equiv \sigma_{\mu\alpha\dot{\alpha}} P^{\alpha\dot{\alpha}}$) $P_\mu P^\mu = m^2$. The constraints (7) and (8) are introduced in the action (3) in order to obtain a definite spin s and the covariant spin projection s_3 whereas the constraint (9) defines the $U(1)$ charge q of the particle. The case of a massive spinless particle has recently been described in terms of a single twistor by using a modified twistor-phase space transform inspired by two-time physics techniques [9].

In the subsection 2.3 we shall see, from the time preservation of the constraints, that secondary constraints do not appear in our model. Thus, the full set of constraints is given by the physical constraints (6)-(9) and by the primary ones (4)-(5).

2.2 Analysis of the primary constraints

If we transform the twelve constraints (4), (5) to equivalent Lorentz-invariant expressions by contracting them with the spinors $\pi_{\alpha k}$ and $\bar{\pi}^k_{\dot{\alpha}}$ and combine the results, the discussion of the constraints is simplified and their splitting into first and second class is clearer. The eight expressions (5) take the form

$$D_k{}^j \equiv \mathcal{D}_k{}^j + s_k{}^j \approx 0\,, \qquad B_k{}^j \equiv \mathcal{B}_k{}^j \approx 0\,,$$

where $\mathcal{D}_k{}^j \equiv i(\pi_{\alpha k} P^{\alpha j} - \bar{P}^{\dot{\alpha}}_k \bar{\pi}^j_{\dot{\alpha}})\,, \qquad \mathcal{B}_k{}^j \equiv i(\pi_{\alpha k} P^{\alpha j} + \bar{P}^{\dot{\alpha}}_k \bar{\pi}^j_{\dot{\alpha}})$.

The first four constraints of (4), after contraction with spinors and using the mass shell constraint, take the form

$$C_k{}^l \equiv \mathcal{P}_k{}^l + m^2 \delta_k{}^l \approx 0 \quad , \qquad \text{where} \quad \mathcal{P}_k{}^l \equiv 4\pi_{\alpha k} P^{\alpha\dot{\beta}} \bar{\pi}_{\dot{\beta}}{}^l\,.$$

We transform the new set of 16 primary constraints, in order to get $SU(2)$ vector and scalar quantities, in the following way

$$A_r = \frac{1}{2} A_i^j (\sigma_r)_j^i \quad , \qquad A_0 = \frac{1}{2} A_i^i \quad , \qquad A = \{B, C, D, P_{(s)}\} \quad ,$$

where $(\sigma_r)_j{}^k$, $r = 1, 2, 3$ are the Pauli matrices. The set of primary constraints takes the following form

$$R_r \equiv P_{(s)r} \approx 0 \, , \qquad R_0 \equiv P_{(s)0} \approx 0 \, , \tag{10}$$
$$D_r \equiv \mathcal{D}_r + s_r \approx 0 \, , \qquad D_0 \equiv \mathcal{D}_0 + s_0 \approx 0 \, , \tag{11}$$
$$B_r \equiv \mathcal{B}_r \approx 0 \, , \qquad B_0 \equiv \mathcal{B}_0 \approx 0 \, , \tag{12}$$
$$C_r \equiv \mathcal{P}_r \approx 0 \, , \qquad C_0 \equiv \mathcal{P}_0 + m^2 \approx 0 \, . \tag{13}$$

Thus, our full set of constraints is described now by the four physical constraints (6)-(9) and by the sixteen primary constraints (10)-(13).

We present now the canonical Poisson brackets (PB) of the coordinates $\mathcal{Q}_L$ and their momenta $\mathcal{P}_L$

$$\{x^\mu, P_\nu\} = \delta^\mu_\nu \, , \qquad \{s_k{}^j, P_{(s)}{}^n{}_l\} = \delta^n_k \delta^j_l \, ,$$
$$\{\pi_{\alpha k}, P^{\beta j}\} = \delta^\beta_\alpha \delta^j_k \, , \qquad \{\bar{\pi}^k_{\dot{\alpha}}, \bar{P}^{\dot{\beta}}_j\} = \delta^{\dot{\beta}}_{\dot{\alpha}} \delta^k_j \, ,$$
$$\{s_0, P_{(s)0}\} = \frac{1}{2} \, , \qquad \{s_r, P_{(s)q}\} = \frac{1}{2} \delta_{rq} \, .$$

Evaluating the corresponding Poisson brackets we see that the three quantities $\mathcal{D}_r$ are the generators of $SO(3)$ and the three quantities $\mathcal{B}_r$ extend the $\mathcal{SO}(3)$ algebra to the Lorentz symmetry $SO(3,1) \simeq sl(2;\mathbb{C})$. We shall call the $\mathcal{D}_r$, $\mathcal{B}_r$ internal symmetry generators.

We can also conclude that the quantities $\mathcal{P}_0$, $\mathcal{P}_r$ extend the internal Lorentz generators $(\mathcal{D}_r, \mathcal{B}_r)$ to an internal Poincaré algebra. The complete set of non-vanishing Poisson brackets between all twenty constraints can be found in [10].

2.3 Time evolution of constraints and their split into first and second class ones

The action (3) is invariant under an arbitrary rescaling on the world line $\tau \to \tau' = \tau'(\tau)$ and the canonical Hamiltonian vanishes $\mathcal{H} = \mathcal{P}_L \dot{\mathcal{Q}}_L - \mathcal{L} = 0$. The total Hamiltonian is given, therefore, by a linear combination of all the constraints,

$$\begin{aligned} \mathcal{H}^C &= \lambda_r^{(D)} D_r + \lambda_0^{(D)} D_0 + \lambda_r^{(B)} B_r + \lambda_0^{(B)} B_0 + \lambda_r^{(C)} C_r + \lambda_0^{(C)} C_0 + \\ &\quad + \lambda_r^{(R)} R_r + \lambda_0^{(R)} R_0 + \lambda^{(T)} T + \lambda^{(S)} S + \lambda^{(S_3)} S_3 + \lambda^{(Q)} Q \, . \end{aligned}$$

Imposing the preservation of all the constraints in time we find that four out of the twenty Lagrange multipliers above are not determined. The four first class constraints associated with them are

$$\mathcal{F} = C_0 + \frac{1}{2}T \simeq 0 \quad , \quad \mathcal{S} = S - 2s_r D_r \simeq 0\,, \tag{14}$$

$$\mathcal{S}_3 = S_3 - D_3 - 2\epsilon_{3rq} s_q R_r \simeq 0 \quad , \quad \mathcal{Q} = Q - D_0 \simeq 0\,. \tag{15}$$

The other 16 constraints can be presented as eight pairs of canonically conjugated second class constraints ($D_r \Leftrightarrow R_r\,,\ D_0 \Leftrightarrow R_0\,,\ B_r \Leftrightarrow C_r\,,\ B_0 \Leftrightarrow T$).

The subset of constraints D_r, B_r does not close under the PB operation. To solve this we introduce the following linear combination of constraints

$$\begin{aligned} D_r{}' &\equiv D_r - \epsilon_{rpq} s_p R_q = \mathcal{D}_r + s_r - \epsilon_{rpq} s_p \mathcal{P}_{(s)q} \approx 0\,, \\ B_r{}' &\equiv B_r + \frac{i}{2m^2}\epsilon_{rpq} s_p C_q = \mathcal{B}_r + \frac{i}{2m^2}\epsilon_{rpq} s_p \mathcal{P}_q \approx 0\,. \end{aligned}$$

Now, their Poisson brackets vanish on the surface of the constraints.

3 Solving the second class constraints

Due to the resolution form[3] of the first four pairs of second class constraints we can exclude [10] the variables s_r, s_0 and their momenta $P_{(s)r}$, $P_{(s)0}$ by putting instead

$$s_r = -\mathcal{D}_r\,, \qquad s_0 = -D_0\,, \qquad P_{(s)r} = 0\,, \qquad P_{(s)0} = 0\,.$$

In order to put the next three pairs of second class constraints in the resolution form we perform a canonical transformation[4]

$$(x^\mu\,; P_\mu)\,,\ (\pi_{\alpha k}\,; P^{\alpha k})\,,\ (\bar{\pi}^k_{\dot\alpha}\,; \bar{P}^{\dot\alpha}_k)\ \Leftrightarrow\ (\widetilde{x}_0, \widetilde{x}_r\,; \mathcal{P}_0, \mathcal{P}_r)\,,\ (\pi'_{\alpha k}\,; \mathcal{P}^{\alpha k})\,,\ (\bar{\pi}'^k_{\dot\alpha}\,; \bar{\mathcal{P}}^{\dot\alpha}_k)\,.$$

[3] A pair of constraints $A \approx 0, B \approx 0$ have the *resolution form* in the phase space (x_i, p_i) $i = 1, \ldots, N$ if they have the form given by the following formulae:

$$A = x_1 - f(x_r, p_r) \approx 0 \qquad B = p_1 \approx 0 \quad (r = 2, 3, \ldots, N)$$

This form of the constraints was considered by Dirac [11]. In such a case the Dirac brackets are identical with the canonical PB.

[4] The generating function of this canonical transformation has the form

$$F(P^\mu, \pi_{\alpha k}, \bar{\pi}^k_{\dot\beta}; \widetilde{x}_0, \widetilde{x}_r, \mathcal{P}^{\alpha k}, \bar{\mathcal{P}}^{\dot\alpha}_k) = \tag{16}$$

$$= -[\pi_{\alpha k}\sigma^{\alpha\dot\beta}_\mu \bar{\pi}^k_{\dot\beta} P^\mu]\widetilde{x}_0 + [(\tau_r)_j{}^k \pi_{\alpha k}\sigma^{\alpha\dot\beta}_\mu \bar{\pi}^j_{\dot\beta} P^\mu]\widetilde{x}_r + \pi_{\alpha k}\mathcal{P}^{\alpha k} + \bar{\pi}^k_{\dot\alpha}\bar{\mathcal{P}}^{\dot\alpha}_k\,. \tag{17}$$

Now, we can exclude the variables $\widetilde{x}_r$ and $\mathcal{P}_r$ by setting

$$\widetilde{x}_r = -\frac{i}{\mathcal{P}_0}\mathcal{B}_r\,, \qquad \mathcal{P}_r = 0\,.$$

The only two remaining second class constraints have the form

$$T = 4f\overline{f} - m^2 = 0\,, \qquad B_0 = \mathcal{B}_0 - i\widetilde{x}_0\mathcal{P}_0 = 0\,. \tag{18}$$

Subsequently, we introduce the Dirac brackets (DB) as follows

$$\{y,y'\}_D = \{y,y'\} + \{y,B_0\}\frac{i}{2(T+m^2)}\{T,y'\} - \{y,T\}\frac{i}{2(T+m^2)}\{B_0,y'\}\,,$$

We observe that the first relation of (18) reduces one spinorial degree of freedom, *i.e.* we are left with seven unconstrained spinorial coordinates.

4 First quantization and solution of the first class constraints

4.1 First class constraints

After taking into account all the sixteen second class constraints there remain eighteen phase space variables, namely,

$$\widetilde{x}_0\,,\ \mathcal{P}_0 \quad ; \qquad \pi_{\alpha k}\,,\ \mathcal{P}^{\alpha k}; \qquad \bar{\pi}^k_{\dot{\alpha}}\,,\ \bar{\mathcal{P}}^{\dot{\alpha}}_k \quad ,$$

which are constrained by the two algebraic relations (18). After performing the quantization of the canonical Dirac brackets $\{y,y'\}_D \to \frac{1}{i}[\hat{y},\hat{y}']$ (we put $\hbar = 1$) one obtains the corresponding commutation relations, where we use the '*qp*-ordering'.

The sixteen independent degrees of freedom described by the variables (4.1) are additionally restricted by the four first class constraints (14)-(15). These can be written in the form

$$\begin{aligned} \mathcal{P}_0 + m^2 &\approx 0\,, & (19)\\ \mathcal{D}_r\mathcal{D}_r - s(s+1) &\approx 0\,, & (20)\\ \mathcal{D}_3 + m_3 &\approx 0\,, & (21)\\ \mathcal{D}_0 + q &\approx 0\,, & (22) \end{aligned}$$

where the numerical values of m, s, m_3 and q describe mass, spin, spin projection and an internal Abelian (electric) charge.

4.2 Covariant solution of the constraints

We take the Schrödinger realization of the quantized variables (4.1) on the commuting generalized coordinate space $(\widetilde{x}_0, \pi_{\alpha j}, \overline{\pi}^j_{\dot{\alpha}})$. The generalized momenta $(\mathcal{P}_0, \mathcal{P}^{\beta j}, \overline{\mathcal{P}}^{\dot{\beta}}{}_j)$ have the following differential realizations:

$$\mathcal{P}_0 = -i\frac{\partial}{\partial \widetilde{x}_0},$$

$$\mathcal{P}^{\beta j} = -i\frac{\partial}{\partial \pi_{\beta j}} + i\frac{f}{m^2}\pi^{\beta j}\left(\pi_{\alpha k}\frac{\partial}{\partial \pi_{\alpha k}} + \bar{\pi}^k_{\dot{\alpha}}\frac{\partial}{\partial \bar{\pi}^k_{\dot{\alpha}}} - 2\widetilde{x}_0\frac{\partial}{\partial \widetilde{x}_0}\right),$$

$$\bar{\mathcal{P}}^{\dot{\beta}}_j = -i\frac{\partial}{\partial \bar{\pi}^j_{\dot{\beta}}} - i\frac{\bar{f}}{m^2}\bar{\pi}^{\dot{\beta}}_j\left(\pi_{\alpha k}\frac{\partial}{\partial \pi_{\alpha k}} + \bar{\pi}^k_{\dot{\alpha}}\frac{\partial}{\partial \bar{\pi}^k_{\dot{\alpha}}} - 2\widetilde{x}_0\frac{\partial}{\partial \widetilde{x}_0}\right).$$

The wavefunction has the following coordinate dependence

$$\Psi(\widetilde{x}_0, \pi_{\alpha k}, \bar{\pi}^k_{\dot{\alpha}}).$$

We shall solve the four wave equations consecutively:

i) Mass shell constraint (19).

The general solution of the constraint (19) is

$$\Psi(\widetilde{x}_0, \pi_{\alpha k}, \bar{\pi}^k_{\dot{\alpha}}) = e^{-im^2\widetilde{x}_0}\,\Phi(\pi_{\alpha k}, \bar{\pi}^k_{\dot{\alpha}}). \tag{23}$$

One can show [10] that the invariant time coordinate $\widetilde{\tau} = m^2\widetilde{x}_0 = P^\mu x_\mu$. Therefore, the exponent in the wavefunction (23) has the standard form of a plane wave, where the four-momentum P_μ is composite in terms of spinors.

ii) Normalized spinors and electric charge.

The eight (real) variables $\pi_{\alpha k}$, $\bar{\pi}^k_{\dot{\alpha}}$ define two variables f, $\bar{f}$ as follows

$$\pi^{\alpha k}\pi_{\alpha k} = 2\bar{f}, \qquad \bar{\pi}_{\dot{\alpha}k}\bar{\pi}^{\dot{\alpha}k} = 2f$$

and the remaining six degrees of freedom can be described by the normalized spinors $u_{\alpha i} = \left(\frac{\bar{f}}{f}\right)^{-1/4}\pi_{\alpha i}$, $\bar{u}^i_{\dot{\alpha}} = \overline{(u_{\alpha i})} = \left(\frac{\bar{f}}{f}\right)^{1/4}\bar{\pi}^i_{\dot{\alpha}}$.

Due to the first constraint of (18), the modulus of f is given by the mass parameter $|f| = \frac{m}{2}$ and the variable $y \in S^1$, $y \equiv \frac{\bar{f}}{f}$, defines the phase of f which will be eliminated by the constraint (22).

Using the variables y, $u_{\alpha i}$, $\bar{u}^i_{\dot{\alpha}}$ the first class constraint (22) has the solution

$$\Phi_m(y, u_{\alpha k}, \bar{u}^k_{\dot{\alpha}}) = y^{-q/2}\tilde{\Phi}_m(u_{\alpha k}, \bar{u}^k_{\dot{\alpha}}).$$

iii) Spin content.

Let us find now the solution of the constraints (20), (21) for the function $\tilde{\Phi}(u_{\alpha k}, \bar{u}^k_{\dot{\alpha}})$ using the polynomial expansion in spinor variables

$$\tilde{\Phi}(u_{\alpha k}, \bar{u}^k_{\dot{\alpha}}) = \sum_{k,n=0}^{\infty} \frac{1}{k!\, n!}\, u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}^{j_1}_{\dot{\beta}_1} \ldots \bar{u}^{j_n}_{\dot{\beta}_n}\, \phi^{\alpha_1 \ldots \alpha_k \dot{\beta}_1 \ldots \dot{\beta}_n}{}^{i_1 \ldots i_k}_{j_1 \ldots j_n}(P_\mu)\,.$$

The solution of Eq. (20) is

$$\tilde{\Phi}(u_{\alpha k}, \bar{u}^k_{\dot{\alpha}}) = \sum_{k,n;\, k+n=2s} \frac{1}{k!\, n!}\, u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}^{j_1}_{\dot{\beta}_1} \ldots \bar{u}^{j_n}_{\dot{\beta}_n}\, \phi^{\alpha_1 \ldots \alpha_k \dot{\beta}_1 \ldots \dot{\beta}_n}{}^{i_1 \ldots i_k}_{j_1 \ldots j_n}(P_\mu)\,,$$

where in this expansion only spinorial polynomials of order $2s$ ($k = k_1 + k_2$, $n = n_1 + n_2$) are present, $k + n = 2s$, where k_i ($i = 1, 2$) denotes the number of spinors u^α_i, and n_i ($i = 1, 2$) the number of spinors $\overline{u}^{\dot{\alpha}}_i$.

In order to describe covariant projection of the spin, given by the eigenvalue equation (21), we observe that

$$\begin{aligned} (\mathcal{D}_3 + m_3)(u_{\alpha_1 i_1} \cdots u_{\alpha_k i_k} \overline{u}^{j_1}_{\dot{\beta}_1} \cdots \overline{u}^{j_n}_{\dot{\beta}_n}) = 0 \ , \\ \text{if} \qquad m_3 = \tfrac{1}{2}(n_1 - n_2 - (k_1 - k_2))\,. \end{aligned}$$

A more detailed discussion of the procedure used to obtain the solutions, the link between the variables u^α_i and the spinorial Lorentz harmonics can be found in [10]. Although we are going to present here the example for the spin 1 wavefunctions the simpler spin $\frac{1}{2}$ case can also be found in [5, 10].

5 Example: Spin $s = 1$.

In this case the field $\tilde{\Phi}(u_{\alpha k}, \bar{u}^k_{\dot{\alpha}})$ is

$$\tilde{\Phi}(u_{\alpha k}, \bar{u}^k_{\dot{\alpha}}) = \frac{1}{2}\, u_{\alpha i} u_{\beta j}\, \phi^{\alpha\beta ij} + u_{\alpha i} \bar{u}^j_{\dot{\beta}}\, \phi^{\alpha\dot{\beta}i}{}_j + \frac{1}{2}\, \bar{u}^i_{\dot{\alpha}} \bar{u}^j_{\dot{\beta}}\, \phi^{\dot{\alpha}\dot{\beta}}{}_{ij}\,. \tag{24}$$

Inserting in this expression $u_{\alpha i} = \frac{2}{m} P_{\alpha\dot{\alpha}} \bar{u}^{\dot{\alpha}}_i$, $\bar{u}^i_{\dot{\alpha}} = -\frac{2}{m} P_{\alpha\dot{\alpha}} u^{\alpha i}$ and comparing the result with (24) we obtain the following equations

$$P_{\alpha\dot{\alpha}} \phi^{\dot{\alpha}ij}_{\beta} + \tfrac{m}{2} \phi_{\alpha\beta}{}^{ij} = 0\,, \qquad P^{\dot{\alpha}\alpha} \phi^{\dot{\beta}ij}_{\alpha} + \tfrac{m}{2} \phi^{\dot{\alpha}\dot{\beta}ij} = 0\,, \tag{25}$$

$$\tfrac{1}{2}\,(P_{\alpha\dot{\alpha}} \phi^{\dot{\alpha}\dot{\beta}ij} + P^{\dot{\beta}\beta} \phi_{\alpha\beta}{}^{ij}) + \tfrac{m}{2} \phi^{\dot{\beta}ij}_{\alpha} = 0\,. \tag{26}$$

The antisymmetric parts of equations (25) provide the transversality condition for fields $\phi^{\alpha\dot{\beta}j}{}_i$

$$P_{\alpha\dot{\beta}} \phi^{\alpha\dot{\beta}ij} = 0\,. \tag{27}$$

Using $P_{\alpha\dot{\beta}} P^{\dot{\beta}\beta} = \frac{1}{4} m^2 \delta_\alpha^\beta$ we obtain further

$$P_{\alpha\dot{\alpha}} \phi^{\dot{\alpha}\dot{\beta}ij} + \frac{m}{2} \phi_\alpha^{\dot{\beta}ij} = 0 \,, \qquad P^{\dot{\beta}\beta} \phi_{\alpha\beta}{}^{ij} + \frac{m}{2} \phi_\alpha^{\dot{\beta}ij} = 0 \,. \tag{28}$$

The equations (25)-(28) are the Bargmann-Wigner equations written in a two-spinor notation. One can pass to the four-component Dirac spinor notation if one constructs from the fields $\phi_{\alpha\beta}{}^{ij}$, $\phi^{\dot{\alpha}\dot{\beta}ij}$, $\phi_\alpha{}^{\dot{\beta}ij}$ and $\phi^{\dot{\beta}}{}_\alpha{}^{ij} \equiv \phi_\alpha{}^{\dot{\beta}ij}$ the following Bargmann-Wigner fields $\psi_{ab}{}^{ij} = \begin{pmatrix} \phi_{\alpha b}{}^{ij} \\ \phi^{\dot{\alpha}}{}_b{}^{ij} \end{pmatrix} = \begin{pmatrix} \phi_{a\beta}{}^{ij} \\ \phi_a{}^{\dot{\beta}ij} \end{pmatrix}$, with double Dirac indices $a, b = 1, 2, 3, 4$. Since $\phi_{\alpha\beta}{}^{ij} = \phi_{\beta\alpha}{}^{ij}$, $\phi^{\dot{\alpha}\dot{\beta}ij} = \phi^{\dot{\beta}\dot{\alpha}ij}$ the fields $\psi_{ab}{}^{ij}$ are symmetric, $\psi_{ab}{}^{ij} = \psi_{ba}{}^{ij}$. Due to the equations (25)-(28) the fields $\psi_{ab}{}^{ij}$ satisfy the Bargmann-Wigner-Dirac equation for massive spin 1 fields: $P^\mu \, \gamma_{\mu a}{}^b \, \psi_{bc}{}^{ij} + m \psi_{ac}{}^{ij} = 0$.

We obtain Proca fields if we define the fields

$$A_\mu{}^{ij} = \sigma_{\mu}{}_{\dot{\beta}}^{\alpha} \phi_\alpha{}^{\dot{\beta}ij} \quad , \qquad F_{\mu\nu}{}^{ij} = m (\sigma_{\mu\nu}{}_\beta^\alpha \phi_\alpha^{\beta ij} + \bar{\sigma}_{\mu\nu}{}_{\dot{\beta}}^{\dot{\alpha}} \phi_{\dot{\alpha}}^{\dot{\beta}ij}) \,. \tag{29}$$

Inserting (29) into the equations (25)-(28) we obtain the spin 1 Proca equations

$$P^\mu A_\mu{}^{ij} = 0 \,, \quad P_\mu A_\nu{}^{ij} - P_\nu A_\mu{}^{ij} = F_{\mu\nu}{}^{ij} \,, \quad P^\mu F_{\mu\nu}{}^{ij} - m^2 A_\nu{}^{ij} = 0 \,, \tag{30}$$

as well as the identity $P_{[\mu} F_{\nu\lambda]}{}^{ij} = 0$.

We obtained three complex fields (internal $SU(2)$-triplet) with spin $s = 1$. On the function (24) we impose the reality condition $\widetilde{\Phi} = \overline{\widetilde{\Phi}}$ which gives $\phi^{\dot{\alpha}\dot{\beta}}{}_{ij} = \bar{\phi}^{\dot{\alpha}\dot{\beta}}{}_{ij} = \overline{(\phi^{\alpha\beta ij})}$, $\phi^{\alpha\dot{\beta}i}{}_j = \overline{(\phi^{\beta\dot{\alpha}j}{}_i)}$. The second set of relations in (30) can be written down as the following $SU(2)$-Majorana reality conditions $\psi^{ij}_{cd} (C\gamma_5)_{ca} (C\gamma_5)_{db} = \epsilon^{ik} \epsilon^{jl} \overline{\psi}_{klab}$ and the fields (29) satisfy the reality conditions $\overline{(A_\mu{}^{ij})} = A_{\mu ij}$, $\overline{(F_{\mu\nu}{}^{ij})} = F_{\mu\nu ij}$. These relations define three real vector fields and the corresponding three real field strengths.

6 Conclusions

We have described a classical and first-quantized model of massive relativistic particles with spin based on a hybrid geometry of phase space, with primary spacetime coordinates x_μ and composite four-momenta P_μ expressed in terms of fundamental spinorial variables. We would like to point out that a model for massive particles with spin in an enlarged spacetime derived from two-twistor geometry, with primary both spacetime coordinates and four-momenta P_μ, has been recently described in [3]-[5]. The difference with our approach here consists in the choice of the primary geometric variables which in [12]-[14] contains, besides two-twistor degrees of freedom in mixed twistorial-spacetime formulation, a primary even Weyl spinor [15].[5] In this work all the degrees of freedom describing massive particles with spin and internal charge are derived entirely from the two-twistor geometry.

In order to quantize the classical system we have introduced a complete set of commuting observables, which determine the generalized coordinates of the wavefunction. In our case the set of commuting generalized coordinates does not contain all the spacetime coordinates, because in our geometric framework they do not commute. As a result, only the Lorentz-invariant projection $m^2\widetilde{x}_0 = x_\mu P^\mu$ can be included into the quantum-mechanical commuting coordinates. In such a way we are allowed to use the plane waves $e^{ix_\mu P^\mu}$ as describing the spacetime dependence of the wavefunction. We conclude, therefore, that although in our framework the spacetime coordinates of spinning massive particles are non-commutative, we are able to obtain the standard plane wave solutions.

Acknowledgements

Two of us (S.F. and C.M.E.) would like to thank the organizers of the *22nd Max Born Symposium* for its very pleasant atmosphere. The work of S.F. was supported in part by the RFBR grant 06-02-16684, the grant INTAS-05-7928 and the grants from Bogoliubov-Infeld and Heisenberg-Landau programs. Two of the authors (A.F. and J.L) would like to acknowledge the support by KBN grant 1 P03B 01828. The work of C.M.E. has been supported by research grants from the Spanish Ministerio de Educación y Ciencia (FIS2005-02761 and EU FEDER funds) , the Generalitat Valenciana and by the EU network MRTN-CT-2004-005104 (Forces Universe). One of us (C.M.E.) wishes to thank Prof. Siegel and the members of the YITP of Stony Brook for the kind hospitality extended to him. Discussions with José de Azcárraga are also acknowledged.

References

[1] L.P. Hughston, *Twistors and Particles*, Lecture Notes in Physics **97**, Springer Verlag, Berlin (1979)

[2] T. Shirafuji, Progr. Theor. Phys. **70**, 18 (1983)

[5] It is worth mentioning that the fixing of spin in [15] leads to violation of even "supersymmetry". The models with a bosonic counterpart of supersymmetry describe higher spin particle [16, 17] and higher spin superparticle [18].

[3] A. Bette, J.A. de Azcárraga, J. Lukierski and C. Miquel-Espanya, Phys. Lett. **B595**, 491-497 (2004)

[4] A. Bette, J. Lukierski and C. Miquel-Espanya, Proceedings of the XIXth Max Born Symposium, September 2004, AIP Conference Series **767**, p. 44 (2005)

[5] J.A. de Azcárraga, A.Frydryszak, J. Lukierski and C. Miquel-Espanya, Phys. Rev. **D73**, 105011 (2006) [hep-th/0510161]

[6] M. Vasiliev, Int. J. Mod. Phys. **D5**, 763 (1996); Phys. Rev. **D66**, 066006 (2002)

[7] E. Sezgin, J. Enquist and P. Sundell, Class. Quant. Grav. **19**, 6175 (2002)

[8] D. Sorokin, in Proceedings of XIXth Max Born Symposium, September 2004, AIP Conference Seriez **767**, 172 (2005)

[9] I. Bars and M. Picón, Phys. Rev. **D73**, 064002 (2006) [hep-th/0512091]

[10] S. Fedoruk, A. Frydryszak, J. Lukierski and C. Miquel-Espanya, Int. J. Mod. Phys. **A21**, 4137-4160 (2006) [hep-th/0510266]

[11] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva Univ., New York 1964

[12] S. Fedoruk and V.G. Zima, Nucl. Phys. **B102-103** (Proc. Suppl.) 233 (2001)

[13] S. Fedoruk and V.G. Zima, J. Kharkov Univ. **585**, 39 (2003), hep-th/0308154;
J. Kharkov Univ. **628**, 89 (2004) (in Russian)

[14] S. Fedoruk, in Proceedings of the XIXth Max Born Symposium, September 2004, AIP Conference Series **767**, 106 (2005)

[15] S. Fedoruk and V.G. Zima, JETP Lett. **61**, 251 (1995)

[16] S. Fedoruk and J. Lukierski, Phys.Lett. JETP Lett. **B632**, 371 (2006)

[17] S. Fedoruk and E. Ivanov, Class.Quant.Grav. **23**, 5195 (2006)

[18] S. Fedoruk, E. Ivanov and J. Lukierski, Phys.Lett. **B641**, 226 (2006)